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Minimal Geršgorin tensor eigenvalue inclusion set and its numerical approximation

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Abstract

For a complex tensor $\mathcal{A}$, Minimal Geršgorin tensor eigenvalue inclusion set of $\mathcal{A}$ is presented, and its sufficient and necessary condition is given. Furthermore, we study its boundary by the spectrums of the equimodular set and the extended equimodular set for $\mathcal{A}$. Lastly, for an irreducible tensor, a numerical approximation to Minimal Geršgorin tensor eigenvalue inclusion set is given.

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1. Introduction

Let $N = \{1, 2, \ldots, n\}$. For a complex (real) order m dimension n tensor $\mathcal{A} = (a_{i_1\cdots i_m})$ (written $\mathcal{A} \in \mathbb{C}^{[m,n]}$ ($\mathbb{R}^{[m,n]}$), respectively), where

$$a_{i_1\cdots i_m} \in \mathbb{C}\ (\mathbb{R}),\ i_j = 1, \ldots, n,\ j = 1, \ldots, m.$$

we call a complex number λ an eigenvalue of $\mathcal{A}$ and a nonzero complex vector x an eigenvector of $\mathcal{A}$ associated with λ, if

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}, \tag{1.1}$$

where $\mathcal{A}x^{m-1}$ and $x^{[m-1]}$ are vectors, whose ith components are

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\ldots,i_m \in N} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m}$$

and

$$(x^{[m-1]})_i = x_i^{m-1},$$

respectively. Note that there are other definitions of eigenvalue and eigenvectors, such as, H-eigenvalue, D-eigenvalue and Z-eigenvalue; see [15, 18, 19]. Obviously, the definition of eigenvalue for matrices follows from the case $m = 2$.

Tensor eigenvalues and eigenvectors have received much attention recently in the literatures [1, 3, 9, 10, 13, 15, 16, 17, 22]. Many important results on the eigenvalue problem of matrices have been successfully extended to higher order tensors; see [1, 2, 12, 13, 15, 16, 18, 23, 24]. In [15], Qi generalized Geršgorin eigenvalue inclusion theorem from matrices to real supersymmetric tensors, which can be easily extended to generic tensors; see [23].

Theorem 1.1. [15] *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$ and $\sigma(\mathcal{A})$ be the spectrum of $\mathcal{A}$, that is,*

$$\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} : \mathcal{A}x^{m-1} = \lambda x^{[m-1]},\ x \in \mathbb{C}^n \backslash \{0\}\}.$$

Then

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) = \bigcup_{i \in N} \Gamma_i(\mathcal{A}), \tag{1.2}$$

where

$$\Gamma_i(\mathcal{A}) = \left\{ z \in \mathbb{C} : |z - a_{i\cdots i}| \leq r_i(\mathcal{A}) = \sum_{\substack{i_2,\ldots,i_m \in N, \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \right\}$$

and

$$\delta_{i_1 i_2\cdots i_m} = \begin{cases} \delta_{i_1 i_2\cdots i_m} = 1, & if\ i_1 = \cdots = i_m, \\ 0, & otherwise. \end{cases}$$

It is easy to see that when $m = 2$, Theorem 1.1 reduces to the well known Geršgorin eigenvalue inclusion theorem of matrices [7, 21]. Here, we call $\Gamma_i(\mathcal{A})$ the i-th Geršgorin tensor eigenvalue inclusion set. Note that $\Gamma_i(\mathcal{A})$ is a closed set in the complex plane $\mathbb{C}$, Hence, $\Gamma(\mathcal{A})$, which consists of the n sets $\Gamma_i(\mathcal{A})$, is also closed and bounded in $\mathbb{C}$.

Also in [15], Qi obtain another interesting result on Geršgorin tensor eigenvalue inclusion set $\Gamma(\mathcal{A})$.

Theorem 1.2. [15] *If $\Gamma_i(\mathcal{A})$ is disjoint with the other $\Gamma_j(\mathcal{A})$, $j \neq i$, then there are exactly $(m-1)^{n-1}$ eigenvalues which lie in $\Gamma_i(\mathcal{A})$. Furthermore, if all of $\Gamma_i(\mathcal{A})$, $i = l_1, l_2, \ldots, l_k$ are connected but disjoint with the other $\Gamma_j(\mathcal{A})$, $j \neq i$, then there are exactly $k(m-1)^{n-1}$ eigenvalues which lie in $\bigcup_{i=l_1,l_2,\ldots,l_k} \Gamma_i(\mathcal{A})$.*

In this paper, we also focus on Geršgorin tensor eigenvalue inclusion set. And we present Minimal Geršgorin tensor eigenvalue inclusion set, give its sufficient and necessary condition, and research its boundary. For an irreducible tensor, we give a set which approximates to Minimal Geršgorin tensor eigenvalue inclusion set.

2. Minimal Geršgorin tensor eigenvalue inclusion set

In this section, we present Minimal Geršgorin tensor eigenvalue inclusion set and study its characteristic. First, a lemma is given.

Lemma 2.1. [23] *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$ and $D = diag(d_1, d_2, \ldots, d_n)$ be a diagonal nonsingular matrix. If*

$$\mathcal{B} = (b_{i_1\cdots i_m}) = \mathcal{A}D^{-(m-1)} \overbrace{DD\cdots D}^{m-1},$$

where

$$b_{i_1\cdots i_m} = d_{i_1}^{-(m-1)} a_{i_1 i_2\cdots i_m} d_{i_2} \cdots d_{i_m},\ i_1, \ldots, i_m \in N,$$

then $\mathcal{A}$, $\mathcal{B}$ have the same eigenvalues.

From Lemma 2.1, we obtain the following tensor eigenvalue inclusion set.

Theorem 2.1. *Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ and $x = (x_1, x_2, \ldots, x_n)^T$ be an entrywise positive vector, i.e., $x = (x_1, x_2, \ldots, x_n)^T > 0$. Then*

$$\sigma(\mathcal{A}) \subseteq \Gamma^{r^x}(\mathcal{A}) = \bigcup_{i \in N} \Gamma_i^{r^x}(\mathcal{A}), \tag{2.1}$$

where

$$\Gamma_i^{r^x}(\mathcal{A}) = \left\{ z \in \mathbb{C} : |z - a_{i \cdots i}| \leq r_i^x(\mathcal{A}) = \sum_{\substack{i_2, \ldots, i_m \in N, \\ \delta_{i i_2 \cdots i_m} = 0}} \frac{|a_{i i_2 \cdots i_m}| x_{i_2} \cdots x_{i_m}}{x_i^{m-1}} \right\}.$$

Furthermore,

$$\sigma(\mathcal{A}) \subseteq \bigcap_{x > 0} \Gamma^{r^x}(\mathcal{A}). \tag{2.2}$$

Proof. Let $X = diag(x_1, x_2, \ldots, x_n)$ and $\mathcal{B} = \mathcal{A} X^{-(m-1)} \overbrace{X X \cdots X}^{m-1} = (b_{i_1 \cdots i_m})$. It is obvious that X is nonsingular and $\sigma(\mathcal{B}) = \sigma(\mathcal{A})$ from Lemma 2.1. From Theorem 1.1, we have

$$\sigma(\mathcal{B}) \subseteq \Gamma(\mathcal{B}) = \bigcup_{i \in N} \Gamma_i(\mathcal{B}),$$

where

$$\Gamma_i(\mathcal{B}) = \left\{ z \in \mathbb{C} : |z - b_{i \cdots i}| \leq r_i(\mathcal{B}) = \sum_{\substack{i_2, \ldots, i_m \in N, \\ \delta_{i i_2 \ldots i_m} = 0}} |b_{i i_2 \cdots i_m}| \right\}$$

Note that $b_{i \cdots i} = a_{i \cdots i}$ and $r_i(\mathcal{B}) = r_i^x(\mathcal{A})$. Hence, $\Gamma_i(\mathcal{B}) = \Gamma_i^{r^x}(\mathcal{A})$, consequently, $\Gamma(\mathcal{B}) = \Gamma^{r^x}(\mathcal{A})$. Therefore,

$$\sigma(\mathcal{A}) = \sigma(\mathcal{B}) \subseteq \Gamma(\mathcal{B}) = \Gamma^{r^x}(\mathcal{A}).$$

Furthermore, for any $x > 0$, $x \in \mathbb{R}^n$, (2.1) also holds. Hence, (2.2) follows.

The set $\bigcap_{x>0} \Gamma^{r^x}(\mathcal{A})$ in Theorem 2.1 is of interest theoretically because it provides a set, containing all the eigenvalues of $\mathcal{A}$, which is called Minimal Geršgorin tensor eigenvalue inclusion set defined as follows.

Definition 2.1. *Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. Then*

$$\Gamma^R(\mathcal{A}) = \bigcap_{\substack{x \in \mathbb{R}^n, \\ x > 0}} \Gamma^{r^x}(\mathcal{A}) \tag{2.3}$$

is Minimal Geršgorin tensor eigenvalue inclusion set of $\mathcal{A}$, related to the collection of all weighted sums, $r_i^x(\mathcal{A})$, where $x = (x_1, x_2, \ldots, x_n)^T > 0$.

Next, a sufficient and necessary condition for the elements belonging to Minimal Geršgorin tensor eigenvalue inclusion set is provided. Before that, we give some results involving the $Perron-Frobenius$ theory of nonnegative tensors [23]. Given a real tensor $\mathcal{A}$, we call $\mathcal{A}$ nonnegative, denoted by $\mathcal{A} \geq 0$, if every of its entries is nonnegative.

Definition 2.2. *[1, Definition 2.1] A tensor $\mathcal{A} = (a_{i_1\cdots,i_m}) \in \mathbb{C}^{[m,n]}$ is called reducible, if there exists a nonempty proper index subset $I \subset N$ such that*

$$|a_{i_1\cdots,i_m}| = 0,\ \forall i_1 \in I,\ \forall i_2,\ldots,i_m \notin I.$$

If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible.

Lemma 2.2. [23] *If $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is nonnegative, then the spectral radius*

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}$$

is an eigenvalue with a nonnegative eigenvector $x \neq 0$ corresponding to it. Moreover, if $\mathcal{A}$ is irreducible, then $\rho(\mathcal{A}) > 0$ and x is positive.

Lemma 2.3. [23, Theorem 5.3] *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be nonnegative. Then*

$$\rho(\mathcal{A}) = \max_{x\geq 0, x\neq 0}\min_{x_i>0}\frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}} = \min_{x\geq 0, x\neq 0}\max_{x_i>0}\frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}}.$$

Remark 2.1. *The first equality is only given in Theorem 5.3 of [23]. And similar to the proof of Theorem 5.3 of [23], the second is proved easily.*

From Lemma 2.3, we can obtain the following result.

Corollary 2.1. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be nonnegative. Then*

$$\rho(\mathcal{A}) = \sup_{x>0}\min_{i\in N}\frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}} = \inf_{x>0}\max_{i\in N}\frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}}.$$

From Corollary 2.1, we have the following result.

Lemma 2.4. *Let $\mathcal{A} = (a_{i_1\cdots,i_m}) \in \mathbb{C}^{[m,n]}$ and z be any complex number. And let $\mathcal{B}(z) = (b_{i_1\cdots,i_m}) \in \mathbb{R}^{[m,n]}$, where*

$$b_{i\cdots i} = -|z - a_{i\cdots i}|,\ b_{ii_2\cdots,i_m} = |a_{ii_2\cdots,i_m}|\ for\ i \in N\ and\ \delta_{ii_2\cdots,i_m} = 0.$$

Then $\mathcal{B}(z)$ possesses a real eigenvalue $v(z)$ which has the property that if $\lambda \in \sigma(\mathcal{B}(z))$, then

$$Re(\lambda) \leq v(z).$$

Furthermore,

$$v(z) = \inf_{x>0}\max_{i\in N}\frac{(\mathcal{B}(z)x^{m-1})_i}{x_i^{m-1}}. \tag{2.4}$$

Proof. Let $\mu = \max\limits_{i\in N}|z - a_{i\cdots i}|$, and $\mathcal{C} = (c_{i_1\cdots,i_m}) \in \mathbb{R}^{[m,n]}$, where

$$c_{i\cdots i} = \mu - |z - a_{i\cdots i}|\ and\ c_{ii_2\cdots,i_m} = |a_{ii_2\cdots,i_m}|,\ i \in N,\ \delta_{ii_2\cdots,i_m} = 0. \tag{2.5}$$

Then $B(z) = -\mu\mathcal{I} + \mathcal{C}$, where $\mathcal{C} \geq 0$. From Equality (1.1), it is obvious that $\lambda(\mathcal{C}) \in \sigma(\mathcal{C})$ if and only if $-\mu + \lambda(\mathcal{C}) \in \sigma(\mathcal{B}(z))$. Moreover, From Lemma 2.2, $\rho(\mathcal{C})$ is an eigenvalue of $\mathcal{C}$ with a nonnegative eigenvector. Hence, $-\mu + \rho(\mathcal{C})$ is an eigenvalue of $\mathcal{B}(z)$ with a nonnegative eigenvector. Let

$$v(z) = -\mu + \rho(\mathcal{C}). \tag{2.6}$$

Obviously, $v(z) \in \sigma(\mathcal{B}(z))$ is real. And for any $\lambda \in \sigma(\mathcal{B}(z))$, we have

$$Re(\lambda) = Re(\lambda + \mu - \mu) = Re(\lambda + \mu) - \mu \leq |\lambda + \mu| - \mu \leq \rho(\mathcal{C}) - \mu = v(z).$$

Furthermore, by Corollary 2.1, we have

$$\rho(\mathcal{C}) = \inf_{x>0} \max_{i \in N} \frac{(\mathcal{C}x^{m-1})_i}{x_i^{m-1}}.$$

Then

$$\begin{aligned}
v(z) &= -\mu + \rho(\mathcal{C}) \\
&= -\mu + \inf_{x>0} \max_{i \in N} \frac{(\mathcal{C}x^{m-1})_i}{x_i^{m-1}} \\
&= \inf_{x>0} \max_{i \in N} \frac{((-\mu\mathcal{I} + \mathcal{C})x^{m-1})_i}{x_i^{m-1}} \\
&= \inf_{x>0} \max_{i \in N} \frac{(\mathcal{B}(z)x^{m-1})_i}{x_i^{m-1}}.
\end{aligned}$$

The conclusion follows.

For $v(z)$ in Lemma 2.4, we give the following property.

Proposition 2.1. *Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ and $v(z)$ be defined as Lemma 2.4. Then $v(z)$ is uniformly continuous on $\mathbb{C}$.*

Proof. Let $\mathcal{B}(z) \in \mathbb{R}^{[m,n]}$ be defined as Lemma 2.4. Then, similar to the proof of Theorem 2.2,

$$v(z) = \inf_{x>0} \max_{i \in N} \frac{(\mathcal{B}(z)x^{m-1})_i}{x_i^{m-1}} = \inf_{x>0} \max_{i \in N} \{r_i^x(\mathcal{A}) - |z - a_{i \cdots i}|\}.$$

Note that for any $z,\ \tilde{z} \in \mathbb{C}$ and $x > 0,\ x \in \mathbb{R}^n$,

$$\begin{aligned}
\max_{i \in N} \{r_i^x(\mathcal{A}) - |z - a_{i \cdots i}|\} &= \max_{i \in N} \{r_i^x(\mathcal{A}) - |z - \tilde{z} + \tilde{z} - a_{i \cdots i}|\} \\
&\geq \max_{i \in N} \{r_i^x(\mathcal{A}) - (|z - \tilde{z}| + |\tilde{z} - a_{i \cdots i}|)\} \\
&= \max_{i \in N} \{r_i^x(\mathcal{A}) - |\tilde{z} - a_{i \cdots i}|\} - |z - \tilde{z}|,
\end{aligned}$$

that is,

$$\max_{i \in N} \{r_i^x(\mathcal{A}) - |\tilde{z} - a_{i \cdots i}|\} \leq |z - \tilde{z}| + \max_{i \in N} \{r_i^x(\mathcal{A}) - |z - a_{i \cdots i}|\}.$$

Hence,

$$\begin{aligned}
\inf_{x>0} \max_{i \in N} \{r_i^x(\mathcal{A}) - |\tilde{z} - a_{i \cdots i}|\} &\leq \inf_{x>0} \left\{ |z - \tilde{z}| + \max_{i \in N} \{r_i^x(\mathcal{A}) - |z - a_{i \cdots i}|\} \right\} \\
&= |z - \tilde{z}| + \inf_{x>0} \max_{i \in N} \{r_i^x(\mathcal{A}) - |z - a_{i \cdots i}|\},
\end{aligned}$$

which implies

$$v(\tilde{z}) - v(z) \leq |z - \tilde{z}|.$$

Similarly, we can obtain

$$v(z) - v(\tilde{z}) \leq |z - \tilde{z}|.$$

Therefore,

$$|v(z) - v(\tilde{z})| \leq |z - \tilde{z}|.$$

This implies that $v(z)$ is uniformly continuous on $\mathbb{C}$.

We now establish the sufficient and necessary condition for $\Gamma^R(\mathcal{A})$.

Theorem 2.2. *Let $\mathcal{A}$, $\mathcal{B}(z)$ and $v(z)$ be defined as Lemma 2.4. Then*

$$z \in \Gamma^R(\mathcal{A}) \ if \ and \ only \ if \ v(z) \geq 0. \tag{2.7}$$

Proof. Assume that $z \in \Gamma^R(\mathcal{A})$. From Definition 2.1, we have that for each vector $x > 0$, $z \in \Gamma^{r^x}(\mathcal{A})$. Hence, there is an $i_0 \in N$ such that

$$|z - a_{i_0 \cdots i_0}| \leq r^x_{i_0}(\mathcal{A}),$$

equivalently,

$$r^x_{i_0}(\mathcal{A}) - |z - a_{i_0 \cdots i_0}| \geq 0.$$

Note that for any $i \in N$,

$$\frac{(\mathcal{B}(z)x^{m-1})_i}{x_i^{m-1}} = r^x_i(\mathcal{A}) - |z - a_{i \cdots i}|.$$

Then $\frac{(\mathcal{B}(z)x^{m-1})_{i_0}}{x_{i_0}^{m-1}} \geq 0$, which implies that for each vector $x > 0$,

$$\max_{i \in N} \frac{(\mathcal{B}(z)x^{m-1})_i}{x_i^{m-1}} \geq 0.$$

By Lemma 2.4,

$$v(z) = \inf_{x>0} \max_{i \in N} \frac{(\mathcal{B}(z)x^{m-1})_i}{x_i^{m-1}} \geq 0.$$

Conversely, suppose that $v(z) \geq 0$. From Equality (2.4), then for each vector $x > 0$, there is an $i \in N$ such that

$$0 \leq v(z) \leq \frac{(\mathcal{B}(z)x^{m-1})_i}{x_i^{m-1}} = r^x_i(\mathcal{A}) - |z - a_{i \cdots i}|.$$

Hence, $|z - a_{i \cdots i}| \leq r^x_i(\mathcal{A})$, and then $z \in \Gamma^{r^x}_i(\mathcal{A})$ which implies $z \in \Gamma^{r^x}(\mathcal{A})$. Since this inclusion holds for each vector $x > 0$, we have $z \in \Gamma^R(\mathcal{A})$ from Definition 2.1.

3. Boundary of $\Gamma^R(\mathcal{A})$

In Section 2, a sufficient and necessary condition for the elements belonging to Minimal Geršgorin tensor eigenvalue inclusion set $\Gamma^R(\mathcal{A})$, is given. We in this section focus on the boundary of $\Gamma^R(\mathcal{A})$, and establish relationships between its boundary (see Definition 3.1), the spectrum of the equimodular set for $\mathcal{A}$ (see Definition 3.2), the spectrum of the extended equimodular set for $\mathcal{A}$ (see Definition 3.3) and $\Gamma^R(\mathcal{A})$.

Definition 3.1. *Let $\mathcal{A} \in \mathbb{C}^{[m,n]}$. The boundary of Minimal Geršgorin tensor eigenvalue inclusion set $\Gamma^R(\mathcal{A})$, denoted by $\partial\Gamma^R(\mathcal{A})$, is defined by*

$$\partial\Gamma^R(\mathcal{A}) = \overline{\Gamma^R(\mathcal{A})} \bigcap \overline{(\Gamma^R(\mathcal{A}))'},$$

where $(\Gamma^R(\mathcal{A}))'$ is the complement of $\Gamma^R(\mathcal{A})$, and $\overline{\Gamma^R(\mathcal{A})}$ is the closure of $\Gamma^R(\mathcal{A})$.

Definition 3.2. *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$. *The equimodular set of* $\mathcal{A}$, *denoted by* $\Omega(\mathcal{A})$, *is defined as*

$$\Omega(\mathcal{A}) = \{\mathcal{Q} = (q_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]} : q_{i\cdots i} = a_{i\cdots i}, |q_{ii_2\cdots i_m}| = |a_{ii_2\cdots i_m}|,\ i \in N, \delta_{ii_2\cdots i_m} = 0\}.$$

Definition 3.3. *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$. *The extended equimodular set of* $\mathcal{A}$, *denoted by* $\hat{\Omega}(\mathcal{A})$, *is defined as*

$$\hat{\Omega}(\mathcal{A}) = \{\mathcal{Q} = (q_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]} : q_{i\cdots i} = a_{i\cdots i}, |q_{ii_2\cdots i_m}| \le |a_{ii_2\cdots i_m}|, i \in N, \delta_{ii_2\cdots i_m} = 0\}.$$

A sufficient and necessary condition for the point lying on the boundary of $\Gamma^R(\mathcal{A})$ is given as follows.

Proposition 3.1. *Let* $\mathcal{A}$ *and* $v(z)$ *be defined as Lemma 2.4.* $z \in \partial\Gamma^R(\mathcal{A})$ *if and only if* $v(z) = 0$, *and there is a sequence of* $\{z_j\}_{j=1}^{\infty} \subseteq (\Gamma^R(\mathcal{A}))'$ *(i.e.,* $v(z_j) < 0$ *for all* $j \ge 1$*) such that* $\lim\limits_{j\to\infty} z_j = z$.

Proof. Note that $\Gamma^R(\mathcal{A})$ is a compact set in the complex plane. Hence, from Definition 3.1,

$$\partial\Gamma^R(\mathcal{A}) = \overline{\Gamma^R(\mathcal{A})} \bigcap \overline{(\Gamma^R(\mathcal{A}))'} = \Gamma^R(\mathcal{A}) \bigcap \overline{(\Gamma^R(\mathcal{A}))'}.$$

Furthermore, by Theorem 2.2, we have that

$$z \in \overline{(\Gamma^R(\mathcal{A}))'} \ if\ and\ only\ if\ v(z) \le 0. \tag{3.1}$$

Therefore, if $z \in \partial\Gamma^R(\mathcal{A})$, then, from (2.7) and (3.1), $v(z) = 0$. Note that $(\Gamma^R(\mathcal{A}))'$ is open and unbounded, then, by (3.1) and $z \in \overline{(\Gamma^R(\mathcal{A}))'}$, there is a sequence of $\{z_j\}_{j=1}^{\infty} \subseteq (\Gamma^R(\mathcal{A}))'$ such that $\lim\limits_{j\to\infty} z_j = z$, where $v(z_j) < 0$ for all $j \ge 1$.

Conversely, it is obvious by assumption that $z \in \overline{(\Gamma^R(\mathcal{A}))'}$ and $z \in \Gamma^R(\mathcal{A}) = \overline{\Gamma^R(\mathcal{A})}$, that is,

$$z \in \partial\Gamma^R(\mathcal{A}).$$

The proof is completed.

Proposition 3.2. *For any* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$ *and any* $z \in \mathbb{C}$ *with* $v(z) = 0$, *there is a tensor* $\mathcal{Q} = (q_{i_1\cdots i_m}) \in \Omega(\mathcal{A})$ *for which* z *is an eigenvalue of* $\mathcal{Q}$. *Then*

$$\partial\Gamma^R(\mathcal{A}) \subseteq \sigma(\Omega(\mathcal{A})) \subseteq \sigma(\hat{\Omega}(\mathcal{A})) \subseteq \Gamma^R(\mathcal{A}),$$

where $\sigma(\Omega(\mathcal{A})) = \bigcup\limits_{\mathcal{D}\in\Omega(\mathcal{A})} \sigma(\mathcal{D})$ *and* $\sigma(\hat{\Omega}(\mathcal{A})) = \bigcup\limits_{\mathcal{D}\in\hat{\Omega}(\mathcal{A})} \sigma(\mathcal{D})$.

Proof. From Definition 2.1, 3.2 and 3.3, it is obvious that $\Omega(\mathcal{A}) \subseteq \hat{\Omega}(\mathcal{A})$ and $\sigma(\Omega(\mathcal{A})) \subseteq \sigma(\hat{\Omega}(\mathcal{A})) \subseteq \Gamma^R(\mathcal{A})$. Hence, we next only prove $\partial\Gamma^R(\mathcal{A}) \subseteq \sigma(\Omega(\mathcal{A}))$.

If $z \in \mathbb{C}$ is such that $v(z) = 0$, then, from the proof of Lemma 2.4, there is a vector $y = (y_1, \ldots, y_n)^T \ge 0$, $y \ne 0$ such that $\mathcal{B}(z)y = 0$, where $\mathcal{B}(z)$ is defined as Lemma 2.4. This implies that for any $k \in N$

$$|z - a_{k\cdots k}|y_k^{m-1} = \sum_{\substack{k_2,\ldots,k_m\in N,\\ \delta_{kk_2\cdots k_m}=0}} |a_{kk_2\cdots k_m}|y_{k_2}\cdots y_{k_m}. \tag{3.2}$$

Now, let ψ_k satisfy

$$z - a_{k\cdots k} = |z - a_{k\cdots k}|e^{i\psi_k},\ k \in N,$$

and $\mathcal{Q} = (q_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$, where

$$q_{k\cdots k} = a_{k\cdots k} \textit{ and } q_{kk_2\cdots k_m} = |a_{kk_2\cdots k_m}|e^{i\psi_k},\ k \in N,\ \delta_{kk_2\cdots k_m} = 0.$$

Hence, from Definition 3.2, $\mathcal{Q} \in \Omega(\mathcal{A})$. Moreover, By considering the k-th entry $(\mathcal{Q}y^{m-1})_k$ of $\mathcal{Q}y^{m-1}$, we have that for any $k \in N$,

$$\begin{aligned}
(\mathcal{Q}y^{m-1})_k &= \sum_{k_2,\ldots,k_m \in N} q_{kk_2\cdots k_m} y_{k_2} \cdots y_{k_m} \\
&= q_{k\cdots k} y_k^{m-1} + e^{i\psi_k} \left(\sum_{\substack{k_2,\ldots,k_m \in N, \\ \delta_{kk_2\cdots k_m}=0}} |a_{kk_2\cdots k_m}| y_{k_2} \cdots y_{k_m} \right) \\
&= (z - (z - a_{k\cdots k})) y_k^{m-1} + e^{i\psi_k} \left(\sum_{\substack{k_2,\ldots,k_m \in N, \\ \delta_{kk_2\cdots k_m}=0}} |a_{kk_2\cdots k_m}| y_{k_2} \cdots y_{k_m} \right) \\
&= z y_k^{m-1} + e^{i\psi_k} \left(-|z - a_{k\cdots k}| y_k^{m-1} + \sum_{\substack{k_2,\ldots,k_m \in N, \\ \delta_{kk_2\cdots k_m}=0}} |a_{kk_2\cdots k_m}| y_{k_2} \cdots y_{k_m} \right) \\
&= z y_k^{m-1},\ (\textit{By Equality } (3.2))
\end{aligned}$$

that is,

$$\mathcal{Q}y^{m-1} = zy^{[m-1]}.$$

Note that $y \neq 0$, then z is an eigenvalue of $\mathcal{Q} \in \Omega(\mathcal{A})$, which shows that $v(z) = 0$ implies $z \in \sigma(\Omega(\mathcal{A}))$. From Proposition 3.1, we have that for each point $z \in \partial\Gamma^R(\mathcal{A})$, $v(z) = 0$, consequently, $z \in \sigma(\Omega(\mathcal{A}))$. Hence, $\partial\Gamma^R(\mathcal{A}) \subseteq \sigma(\Omega(\mathcal{A}))$. The proof is completed.

For the sets $\sigma(\hat{\Omega}(\mathcal{A}))$ and $\Gamma^R(\mathcal{A})$, we have the following result.

Proposition 3.3. *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$. *Then*

$$\sigma(\hat{\Omega}(\mathcal{A})) = \Gamma^R(\mathcal{A}).$$

Proof. From Proposition 3.2, we have $\sigma(\hat{\Omega}(\mathcal{A})) \subseteq \Gamma^R(\mathcal{A})$. Hence, we only prove $\Gamma^R(\mathcal{A}) \subseteq \sigma(\hat{\Omega}(\mathcal{A}))$.

Let $z \in \Gamma^R(\mathcal{A})$. Then, from Theorem 2.2, $v(z) \geq 0$. And from the proof of Lemma 2.4, there is a vector $y = (y_1, \ldots, y_n)^T \geq 0$, $y \neq 0$ such that

$$\mathcal{B}(z)y^{m-1} = v(z)y^{[m-1]},$$

where $\mathcal{B}(z)$ is defined as Lemma 2.4. Hence for any $k \in N$,

$$(|z - a_{k\cdots k}| + v(z)) y_k^{m-1} = \sum_{\substack{k_2,\ldots,k_m \in N, \\ \delta_{kk_2\cdots k_m}=0}} |a_{kk_2\cdots k_m}| y_{k_2} \cdots y_{k_m} \tag{3.3}$$

Now, let $\mathcal{Q} = (q_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$, with

$$q_{k\cdots k} = a_{k\cdots k} \ and \ q_{kk_2\cdots k_m} = \mu_k a_{kk_2\cdots k_m}, \ k \in N, \ \delta_{kk_2\cdots k_m} = 0,$$

where

$$\mu_k = \begin{cases} \dfrac{\left(\sum\limits_{\substack{k_2,\ldots,k_m \in N,\\ \delta_{kk_2\cdots k_m}=0}} |a_{kk_2\cdots k_m}|y_{k_2}\cdots y_{k_m}\right) - v(z)y_k^{m-1}}{\sum\limits_{\substack{k_2,\ldots,k_m \in N,\\ \delta_{kk_2\cdots k_m}=0}} |a_{kk_2\cdots k_m}|y_{k_2}\cdots y_{k_m}}, & if \sum\limits_{\substack{k_2,\ldots,k_m \in N,\\ \delta_{kk_2\cdots k_m}=0}} |a_{kk_2\cdots k_m}|y_{k_2}\cdots y_{k_m} > 0, \\ 1, & if \sum\limits_{\substack{k_2,\ldots,k_m \in N,\\ \delta_{kk_2\cdots k_m}=0}} |a_{kk_2\cdots k_m}|y_{k_2}\cdots y_{k_m} = 0. \end{cases}$$

Furthermore, from Equality (3.3) and the fact that both $|z - a_{k\cdots k}|y_k^{m-1} \geq 0$ and $v(z)y_k^{m-1} \geq 0$ hold for any $k \in N$, we easily obtain $0 \leq \mu_k \leq 1$ for any $k \in N$. Hence,

$$\mathcal{Q} \in \hat{\Omega}(\mathcal{A}).$$

For the tensor $\mathcal{Q}$, we have from Equality (3.3) that for any $k \in N$,

$$\begin{aligned} |z - q_{k\cdots k}|y_k^{m-1} &= |z - a_{k\cdots k}|y_k^{m-1} \\ &= \left(\sum_{\substack{k_2,\ldots,k_m \in N,\\ \delta_{kk_2\cdots k_m}=0}} |a_{kk_2\cdots k_m}|y_{k_2}\cdots y_{k_m}\right) - v(z)y_k^{m-1} \\ &= \mu_k \left(\sum_{\substack{k_2,\ldots,k_m \in N,\\ \delta_{kk_2\cdots k_m}=0}} |a_{kk_2\cdots k_m}|y_{k_2}\cdots y_{k_m}\right) \\ &= \sum_{\substack{k_2,\ldots,k_m \in N,\\ \delta_{kk_2\cdots k_m}=0}} |q_{kk_2\cdots k_m}|y_{k_2}\cdots y_{k_m}, \end{aligned}$$

i.e.,

$$|z - q_{k\cdots k}|y_k^{m-1} = \sum_{\substack{k_2,\ldots,k_m \in N,\\ \delta_{kk_2\cdots k_m}=0}} |q_{kk_2\cdots k_m}|y_{k_2}\cdots y_{k_m} \ for \ any \ k \in N.$$

Now, the above expression is exactly of the form of Equality (3.2) in the proof of Proposition 3.2. Hence, similar to the proof of Proposition 3.2, we have that there is a tensor $\mathcal{P} \in \Omega(\mathcal{Q})$ such that $z \in \sigma(\mathcal{P})$. Note that $\mathcal{P} \in \Omega(\mathcal{Q})$ and $\mathcal{Q} \in \hat{\Omega}(\mathcal{A})$. Therefore, $\mathcal{P} \in \hat{\Omega}(\mathcal{A})$, consequently, $z \in \sigma(\hat{\Omega}(\mathcal{A}))$ and $\Gamma^R(\mathcal{A}) \subseteq \sigma(\hat{\Omega}(\mathcal{A}))$. The proof is completed.

From Propositions 3.2 and 3.3, we can obtain the following relationships.

Theorem 3.1. *Let $\mathcal{A} \in \mathbb{C}^{[m,n]}$. Then*

$$\partial\Gamma^R(\mathcal{A}) \subseteq \sigma(\Omega(\mathcal{A})) \subseteq \Gamma^R(\mathcal{A}).$$

Remark 3.1. *From Proposition 3.3, we known that $\sigma(\hat{\Omega}(\mathcal{A}))$ completely fills out $\Gamma^R(\mathcal{A})$, that is, $\sigma(\hat{\Omega}(\mathcal{A})) = \Gamma^R(\mathcal{A})$. And from Theorem 3.1, we known that if $\sigma(\Omega(\mathcal{A}))$ is a proper subset of $\Gamma^R(\mathcal{A})$ with $\sigma(\Omega(\mathcal{A})) \neq \partial\Gamma^R(\mathcal{A})$, then $\sigma(\Omega(\mathcal{A}))$ necessarily have internal boundaries in $\Gamma^R(\mathcal{A})$.*

4. A numerical approximation of Minimal Geršgorin tensor eigenvalue inclusion set

Unlike Geršgorin tensor eigenvalue inclusion set $\Gamma(\mathcal{A})$, or $\Gamma^{r^x}(\mathcal{A})$ of (2.1), Minimal Geršgorin tensor eigenvalue inclusion set $\Gamma^R(\mathcal{A})$ of a complex tensor $\mathcal{A}$ is not easy to determine numerically generally. In this section, for an irreducible tensor $\mathcal{A}$, we give a numerical approximation of $\Gamma^R(\mathcal{A})$, which contains $\Gamma^R(\mathcal{A})$. We now give a lemma, which will be used below.

Lemma 4.1. *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$ be irreducible, and $v(z)$ be defined as Lemma 2.4. Then for each $i \in N$,*

$$v(a_{i\cdots i}) > 0.$$

Furthermore, for each $a_{i\cdots i}$ and for each real θ with $0 \leq \theta < 2\pi$, let $\tilde{\gamma}_i(\theta) > 0$ be the smallest $\gamma > 0$ for which

$$\begin{cases} v(a_{i\cdots i} + \tilde{\gamma}_i(\theta)e^{i\theta}) = 0, \\ \text{and there is a sequence of } \{z_j\}_{j=1}^{\infty} \text{ with} \\ \lim\limits_{j\to\infty} z_j = a_{i\cdots i} + \tilde{\gamma}_i(\theta)e^{i\theta} \text{ such that } v(z_j) < 0 \text{ for all } j \geq 1. \end{cases} \tag{4.1}$$

Then, the complex interval $[a_{i\cdots i} + te^{i\theta}]$, for $0 \leq t \leq \tilde{\gamma}_i(\theta)$, is contained in $\Gamma^R(\mathcal{A})$ for each real $\theta \in [0, 2\pi)$, that is, the set

$$\bigcup_{\theta\ \in[0,2\pi)} [a_{i\cdots i} + te^{i\theta}]_{t=0}^{\tilde{\gamma}_i(\theta)},\ i \in N,$$

is a subset of $\Gamma^R(\mathcal{A})$.

Proof. Let $\mathcal{B}(z) = (b_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be defined as Lemma 2.4, where,

$$b_{i\cdots i} = -|z - a_{i\cdots i}| \text{ and } b_{ii_2\cdots,i_m} = |a_{ii_2\cdots,i_m}|,\ i \in N,\ \delta_{ii_2\cdots,i_m} = 0.$$

And let $\mathcal{C} = (c_{i_1\cdots,i_m}) \in \mathbb{R}^{[m,n]}$, where

$$c_{i\cdots i} = \mu - |z - a_{i\cdots i}| \text{ and } c_{ii_2\cdots,i_m} = |a_{ii_2\cdots,i_m}|,\ i \in N,\ \delta_{ii_2\cdots,i_m} = 0,$$

$\mu = \max\limits_{i\in N} |z - a_{i\cdots i}|$. Then $B(z) = -\mu\mathcal{I} + \mathcal{C}$. Since $\mathcal{A}$ is irreducible, $\mathcal{B}(z)$, also $\mathcal{C}$, is irreducible. Hence, $\mathcal{C}$ is an irreducible nonnegative tensor. From Lemma 2.2, there is a positive eigenvector of $\mathcal{C}$ corresponding to $\rho(\mathcal{C})$. Therefore, similar to the proof of 2.4, there is a positive eigenvector of $\mathcal{B}(z)$ corresponding to $v(z)$. Moreover, by Equality (2.4), that is,

$$v(z) = \inf_{x>0} \max_{i\in N} \frac{(\mathcal{B}(z)x^{m-1})_i}{x_i^{m-1}},$$

we get that for any z, there exists a positive vector y such that for all $j \in N$,

$$v(z) = \frac{(\mathcal{B}(z)y^{m-1})_j}{y_j^{m-1}}. \tag{4.2}$$

Then for any $i \in N$, take $z = a_{i\cdots i}$ and let $x = (x_1, \ldots, x_n)^T > 0$ be such that for any $j \in N$, $v(a_{i\cdots i}) = \frac{(\mathcal{B}(z)x^{m-1})_j}{x_j^{m-1}}$. In particular,

$$v(a_{i\cdots i}) = \frac{(\mathcal{B}(z)x^{m-1})_i}{x_i^{m-1}} = r_i^x(\mathcal{A}) - |a_{i\cdots i} - a_{i\cdots i}| = r_i^x(\mathcal{A}).$$

Since $\mathcal{A}$ is irreducible, we have $r_i^x(\mathcal{A}) > 0$, consequently, $v(a_{i\cdots i}) > 0$.

Furthermore, for each θ with $0 \leq \theta < 2\pi$, $a_{i...i} + te^{i\theta}$, for all $t \geq 0$, is the semi-infinite complex line. Obviously, the function $v(a_{i...i} + te^{i\theta})$ is positive at $t = 0$, is continuous on this line, and is negative outside $\Gamma^R(\mathcal{A})$ from Theorem 2.2. Hence, there is a smallest $\tilde{\gamma}_i(\theta) > 0$ satisfying (4.1). And by Proposition 3.1, we get that $a_{i...i} + \tilde{\gamma}_i(\theta)e^{i\theta} \in \partial\Gamma^R(\mathcal{A})$. This implies that for each real θ, $[a_{i...i}, a_{i...i} + \tilde{\gamma}_i(\theta)e^{i\theta}]$ is a subset of $\Gamma^R(\mathcal{A})$. The conclusion follows.

We now give the following procedure for approximating Minimal Geršgorin tensor eigenvalue inclusion set $\Gamma^R(\mathcal{A})$ for an irreducible tensor $\mathcal{A} = (a_{i_1\cdots,i_m}) \in \mathbb{C}^{[m,n]}$.

Procedure of numerical approximation

Step 1. determine the positive numbers $\{v(a_{j\cdots j})\}_{j\in N}$;

Step 2. determine the largest $\tilde{\gamma}_j(\theta)$ for $0 \leq \theta < 2\pi$, such that $a_{j\cdots j} + \tilde{\gamma}_j(\theta)e^{i\theta} \in \partial\Gamma^R(\mathcal{A})$, i.e.,

$$v(a_{j\cdots j} + \tilde{\gamma}_j(\theta)e^{i\theta}) = 0, \ \textit{with}\ v(a_{j\cdots j} + (\tilde{\gamma}_j(\theta) + \varepsilon)e^{i\theta}) < 0 \tag{4.3}$$

for all sufficiently small $\varepsilon > 0$;

Step 3. take the m points $w_{k_{j_\theta}} = a_{j\cdots j} + \tilde{\gamma}_j(\theta)e^{i\theta} \in \partial\Gamma^R(\mathcal{A})$ for $k_{j_\theta} = 1, 2, \ldots, m$, and detemine the set $\bigcap\limits_{k_{j_\theta}=1}^{m} \Gamma^{w_{k_{j_\theta}}}(\mathcal{A})$ approximating to $\Gamma^R(\mathcal{A})$, where

$$\Gamma^{w_{k_{j_\theta}}}(\mathcal{A}) = \bigcup_{i\in N}\{z \in \mathbb{C} : |z - a_{i\cdots i}| \leq |w_{k_{j_\theta}} - a_{i\cdots i}|\}.$$

Remark 4.1. *(i) To determine $v(a_{j\cdots j})$, we use Equality (2.6) with $z = a_{j\cdots j}$, i.e.,*

$$v(a_{j\cdots j}) = -\mu + \rho(\mathcal{C}),$$

where $\mu = \max\limits_{i\in N} |a_{j\cdots j} - a_{i\cdots i}|$ and $\rho(\mathcal{C})$ is an eigenvalue of the irreducible nonnegative tensor $\mathcal{C}$ defined as (2.5) (note that the irreducibility of $\mathcal{C}$ is deduced by that of $\mathcal{A}$), and the following method for calculating $\rho(\mathcal{C})$ (see [12, 13]), i.e., if for an vector $x^{(0)} > 0$, let $\mathcal{D} = \mathcal{C} + h\mathcal{I}$, where $h > 0$, and let $y^{(0)} = \mathcal{D}\left(x^{(0)}\right)^{m-1}$,

$$\begin{aligned}
x^{(1)} &= \frac{\left(y^{(0)}\right)^{[\frac{1}{m-1}]}}{\|\left(y^{(0)}\right)^{[\frac{1}{m-1}]}\|}, & y^{(1)} &= \mathcal{D}\left(x^{(1)}\right)^{m-1},\\
x^{(2)} &= \frac{\left(y^{(1)}\right)^{[\frac{1}{m-1}]}}{\|\left(y^{(1)}\right)^{[\frac{1}{m-1}]}\|}, & y^{(2)} &= \mathcal{D}\left(x^{(2)}\right)^{m-1},\\
&\vdots & &\vdots\\
x^{(k+1)} &= \frac{\left(y^{(k)}\right)^{[\frac{1}{m-1}]}}{\|\left(y^{(k)}\right)^{[\frac{1}{m-1}]}\|}, & y^{(k+1)} &= \mathcal{D}\left(x^{(k+1)}\right)^{m-1},\ k \geq 2\\
&\vdots & &\vdots
\end{aligned}$$

and let

$$\underline{\lambda}_k = \min_{x_i^{(k)}>0} \frac{\left(y^{(k)}\right)_i}{\left(x_i^{(k)}\right)^{m-1}},\ \overline{\lambda}_k = \max_{x_i^{(k)}>0} \frac{\left(y^{(k)}\right)_i}{\left(x_i^{(k)}\right)^{m-1}},\ k = 1, 2, \ldots,$$

then

$$\underline{\lambda}_1 \leq \underline{\lambda}_2 \leq \cdots \leq \rho(\mathcal{D}) = \rho(\mathcal{C}) + h \leq \cdots \leq \overline{\lambda}_2 \leq \overline{\lambda}_1.$$

moreover,

$$\lim_{k\longrightarrow\infty}\underline{\lambda}_k=\rho(\mathcal{D})=\rho(\mathcal{C})+h=\lim_{k\longrightarrow\infty}\overline{\lambda}_k. \tag{4.4}$$

Hence, we can obtain convergent upper and lower estimates of $v(a_{j\cdots j})$, which do not need great accuracy for graphing purpose, as Example 4.1 shows.

(ii) The numerical estimation of $\tilde{\gamma}_j(\theta)$. From Lemma 4.1, there is $\tilde{\gamma}_j(\theta)$ such that (4.3) holds. Now, let $z=a_{j\cdots j}$ and $\tilde{z}=a_{j\cdots j}+\tilde{\gamma}_j(\theta)e^{\boldsymbol{i}\theta}$, we have from Proposition 2.1 that

$$\tilde{\gamma}_j(\theta)\geq v(a_{j\cdots j})>0.$$

Hence, $v(a_{j\cdots j}+v(a_{j\cdots j})e^{\boldsymbol{i}\theta})\geq 0$. If $v(a_{j\cdots j}+v(a_{j\cdots j})e^{\boldsymbol{i}\theta})=0$, then take

$$\tilde{\gamma}_j(\theta)=v(a_{j\cdots j})$$

for which $v(a_{j\cdots j})$ can be determined by the method of (i), otherwise, $v(a_{j\cdots j}+v(a_{j\cdots j})e^{\boldsymbol{i}\theta})>0$, then we increase the number $v(a_{j\cdots j})$ to $v(a_{j\cdots j})+\Delta$, $\Delta>0$, until $v(a_{j\cdots j}+(v(a_{j\cdots j})+\Delta)e^{\boldsymbol{i}\theta})<0$, and apply a bisection search to the interval $[v(a_{j\cdots j}),\ v(a_{j\cdots j})+\Delta]$ to determine $\tilde{\gamma}_j(\theta)$ satisfying (4.3). Note here that estimates of $\tilde{\gamma}_j(\theta)$ also do not need great accuracy for graphing purpose.

(iii) It is obvious that $w_{k_{j_\theta}}$ in Step 3, $k_{j_\theta}=1,2,\ldots,m$, are not only boundary points of $\Gamma^R(\mathcal{A})$, but boundary points of $\Gamma^{w_{k_{j_\theta}}}(\mathcal{A})$, and that

$$\Gamma^R(\mathcal{A})\subseteq\Gamma^{w_{k_{j_\theta}}}(\mathcal{A})$$

which shows that the larger m is, the better $\Gamma^{w_{k_{j_\theta}}}(\mathcal{A})$ approximates to $\Gamma^R(\mathcal{A})$.

Example 4.1. *Consider the irreducible tensor*

$$\mathcal{A}=[A(1,:,:),A(2,:,:),A(3,:,:)]\in\mathbb{C}^{[3,3]},$$

where

$$A(1,:,:)=\begin{pmatrix}2&0&0\\0&0&1\\0&0&1\end{pmatrix},\ A(2,:,:)=\begin{pmatrix}0&0&0\\0&2&0\\1&0&0\end{pmatrix},\ A(3,:,:)=\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}.$$

We next give a numerical approximation to $\Gamma^R(\mathcal{A})$. By the part (i) of Remark 4.1, we compute $v(a_{iii})$ for $i=1,2,3$, and get

$$v(a_{111})=v(a_{222})=1.62019803,\ v(a_{333})=1.43720383.$$

Furthermore, based on the entries $a_{111}=a_{222}=2$ and $a_{333}=1$, we look for six points

$$\begin{array}{ll} w_1=a_{111}+\tilde{\gamma}_1(0), & w_2=a_{333}-\tilde{\gamma}_3(\pi),\\ w_3=a_{111}+\boldsymbol{i}\,\tilde{\gamma}_1(\frac{\pi}{2}), & w_4=a_{111}-\boldsymbol{i}\,\tilde{\gamma}_1(\frac{3\pi}{2}),\\ w_5=a_{333}+\boldsymbol{i}\,\tilde{\gamma}_3(\frac{\pi}{2}), & w_6=a_{333}-\boldsymbol{i}\,\tilde{\gamma}_3(\frac{3\pi}{2}) \end{array}$$

of $\partial\Gamma^R(\mathcal{A})$, which are found by the method proposed in the part (ii) of Remark 4.1, that is,

$$\begin{array}{lll} w_1=3.62019802, & w_2=-0.43720383, & w_3=2+\boldsymbol{i}1.86790935,\\ w_4=2-\boldsymbol{i}1.86790935, & w_5=1+\boldsymbol{i}1.81661895, & w_6=1-\boldsymbol{i}1.81661895. \end{array}$$

And now $\Gamma(\mathcal{A})$, $\Gamma^{w_1}(\mathcal{A})$, $\Gamma^{w_2}(\mathcal{A})$, $\Gamma^{w_3}(\mathcal{A})$ ($\Gamma^{w_3}(\mathcal{A})=\Gamma^{w_4}(\mathcal{A})$) and $\Gamma^{w_5}(\mathcal{A})$ ($\Gamma^{w_5}(\mathcal{A})=\Gamma^{w_6}(\mathcal{A})$) are given by Figures.1, 2, 3, 4 and 5, respectively. In Figure 6, the set $\left(\bigcap\limits_{k=1}^{6}\Gamma^{w_k}(\mathcal{A})\right)$,

which approximates to $\Gamma^R(\mathcal{A})$*, is shown with the inner boundary, and the boundary of* $\Gamma(\mathcal{A})$ *is shown with the outside. The six points* $\{w_k\}_{k=1}^6$ *are plotted with asterisks. As we can see,* $\left(\bigcap_{k=1}^{6} \Gamma^{w_k}(\mathcal{A})\right) \subset \Gamma(\mathcal{A})$*, that is, the set which approximates to Minimal Geršgorin tensor eigenvalue inclusion set is also contained in Geršgorin tensor eigenvalue inclusion set. Also, it is easy to see that more points of* $\partial\Gamma^R(\mathcal{A})$ *are given, the better* $\Gamma^{w_k}(\mathcal{A})$ *approximates to* $\Gamma^R(\mathcal{A})$*.*

5. Conclusions

In this paper, we present Minimal Geršgorin tensor eigenvalue inclusion set $\Gamma^R(\mathcal{A})$, give a sufficient and necessary condition for $\Gamma^R(\mathcal{A})$ by using the $Perron - Frobenius$ theory of nonnegative tensors, and establish the relationships between $\partial\Gamma^R(\mathcal{A})$, $\sigma(\Omega(\mathcal{A}))$, $\sigma(\hat{\Omega}(\mathcal{A}))$ and $\Gamma^R(\mathcal{A})$, i.e.,

$$\partial\Gamma^R(\mathcal{A}) \subseteq \sigma(\Omega(\mathcal{A})) \subseteq \sigma(\hat{\Omega}(\mathcal{A})) = \Gamma^R(\mathcal{A}).$$

These results obtained are generalizations of the corresponding results of matrices [21] to higher order tensors. In [11], Li et al. provided two new eigenvalue inclusion sets which are contained in Geršgorin eigenvalue inclusion set for tensors. An interesting problem arises: what's the relationship between Minimal Geršgorin tensor eigenvalue inclusion set and the sets in [11]? In the future, we will research this problem.

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